OFFICE OF NAVAL RESEARCH

Contract N7onr-35801

T. O. I.

NR-041-032

Technical Report No. 91

ON THE MOTION OF ANCHOR CHAINS

by

R. Gran Olsson

GRADUATE DIVISION OF APPLIED MATHEMATICS

BROWN UNIVERSITY

PROVIDENCE, R. I.

August, 1953

PREFACE

The non-linear character of the problem stems from the fact that the structure of the moving system changes with time: as the anchor chain is unwound from the windlass, the links change from rotary to rectilinear motion. Similar changes of the structure of a moving one-dimensional continuum occur in a wide variety of problems concerned with the behavior of plastic-rigid beams and frames under dynamic loading. The mathematical techniques developed in the present paper have, in fact, proved valuable in the treatment of such problems. It is for this reason that the present paper is issued as a technical report under Contract N7onr-35801 even though the subject matter of this paper does not fall within the scope of this contract.

E. H. LEE, Chairman
Graduate Division of
Applied Mathematics
Brown University

ON THE MOTION OF ANCHOR CHAINS

By R. Gran Olsson, Providence, R. I.

1. __Introduction.__ In an earlier paper the problem of the motion
of an anchor chain was treated under some idealizing conditions
such as, for instance, neglecting the friction of the anchor
chain, chain holder and windlass and the moment of inertia of the
windlass with respect to the rotating axis [1]. The anchor
chain, however, also moves on inclined planes with different
angles and friction occurs at each point where the chain is
changing its direction. It may be shown that it is possible to
take all these influences into account without making use of
mathematical tools other than those in the paper cited above. In
other words: it is possible by means of simple substitutions to
consider the motion of anchor chains under more real physical
conditions and to get solutions similar to those obtained earlier.
The results may be interpreted in the same way as before.

Before giving the mathematical treatment of the prob-
lem let us consider the different parts of the anchor mechanism
and the conditions which this mechanism must satisfy in order to
be in agreement with the solution given in this paper. The links
of the chain are moving from the chain locker, through the chain
locker pipe, over the chain holder of the windlass and through
the hawse pipe (Fig. 1)[2]. The chains should have a clear lead
to the hawse pipe. The chain holder particularly and the windlass
generally should be kept well lubricated in order to keep the
friction moment small. The position of the chain locker is very

important if the windlass gear is to work in a satisfactory manner. In (Fig. 2) is shown the amount of grip of the chain by the chain holder: to insure this grip it is essential to have the tail end weight of the chain from the windlass down to the chain locker well below. It is also important for the locker to be below the main deck, with a chain locker pipe (Fig. 1) for controlling the lashing cable when the chain is dropped. On merchant marine ships anchor chains are made up in 90 foot lengths, usually coupled together by a bolt type of shackle necessitating extreme-end links of special size on either side of the shackle. The design of the windlass cable holder must accommodate the pitch of common links as well as the different pitches over the shackle connections as shown in (Fig. 2).

The transition from the main deck to the hawse pipe as shown in (Fig. 3) is to eliminate the wear at the hawse pipe; to reduce the frictional force, which has the effect of decreasing the effective force in the chain between the ship and the anchor, the hawse pipe should be given such a shape as to provide a smooth and continuous race for the chain links.

2. Notations. In this paper the same notations as those of Timoshenko and Young will be used [3]; some more mathematical notations are in agreement with those used in textbooks on elliptic integrals and functions [4].

a = acceleration (cm sec^{-2}),

C_0 = constant of integration

e_1, e_2, e_3 = roots of the equation $4s^3 - g_2 s - g_3 = 0$.

F = force of friction (g),

g = acceleration of gravity (cm sec^{-2}),

g_2, g_3 = invariants of the Weierstrass elliptic function

I = moment of inertia of windlass (g cm sec^2),

k = modulus of the Jacobian elliptic function,

ℓ_1 = length of a section of the chain (cm),

m = mass of the system in motion (g cm^{-1} sec^2)

$m_{red} = J/r^2$ = mass of the windlass reduced to the periphery of
the windlass (g cm^{-1} sec^2),

q = weight per unit length of the chain (g cm^{-1}),

r = average value of the windless radius (cm),

S_i = tension in the chain for an arbitrary section (g),

t = time (sec),

u = parameter of the elliptic function ,

v = velocity (cm sec^{-1}),

W_o = weight of the anchor (g)

$W = W_o + gx$ = weight of the anchor and a length x of the chain (g),

x = displacement, length of the chain from a fixed point (cm)

$x_o = W_o/g$ = length of the chain, corresponding to the weight W_o(cm),

α_i = angle of an arbitrary inclined plane

γ = ratio of retarding and accelerating forces

Δ = discriminant of Weierstrass elliptic function,

δ = incriment of a quantity (δx, δq),

ϵ = angular acceleration of the windlass (sec^{-2}),

μ = coefficient of friction,

$\zeta = (W_o + gx + g\Sigma \ell_1 + gm_{red})/W_o$,

φ = angle of rotation of the windlass,

ω = the real half period of the elliptic p-function

$p(u)$ = the elliptic p-function of Weierstrass,

$\zeta(u)$ = the elliptic ζ- function of Weierstrass.

3. Equation of motion and its integration. By using d'Alemberts principle and the principle of virtual displacements we get as equation of motion of the system by an increment δx of displacement x and $\delta\varphi$ of angle φ of rotation [5].

$$(W_o + gx + g \sum_{i=1}^{n} \ell_1 \sin \alpha_1) \delta x - F\delta x - (I\ddot{\varphi} + M_T) \delta\varphi$$

$$- \frac{1}{g} \frac{d}{dt} \left[(W_o + qx + q\sum\ell_1) v \right] \delta x = 0. \tag{1}$$

As will be shown at the end of this paper it is not necessary to make a more exact consideration of the friction than notating it by a constant force F.

Because of the fixed connection of anchor chain and windlass the kinematic condition yields

$$\delta x = r\delta\varphi \tag{2}$$

and for the velocities and accelerations

$$\dot{x} = v = r\dot{\varphi} , \quad \ddot{x} = a = r\ddot{\varphi} , \tag{2a}$$

respectively.

The term $I\ddot{\varphi} \delta\varphi$ may be written

$$I \frac{\ddot{x}}{r^2} \delta x = m_{red} \frac{dv}{dt} \delta x, \tag{3}$$

where $m_{red} = I/r^2$ is the moment of inertia reduced to the radius r of the windlass. This is added to the last term of Eq.(1). In order to obtain a total differential the expression

$$g \Sigma \ell_i (1 - \sin \alpha_i) + gm_{red}$$

is added to the two first terms of Eq.(1), giving the following equation

$$W_o + qx + q\Sigma\ell_i + gm_{red} - [F + \frac{M_F}{r} + q\Sigma\ell_i (1 - \sin \alpha_i) + gm_{red}]$$

$$= \frac{1}{g} \frac{d}{dt} [(W_o + qx + q\Sigma\ell_i + gm_{red}) v]. \tag{1a}$$

Multiplying by

$$(W_o + qx + q\Sigma\ell_i + gm_{red}) v dt$$

and integrating, we obtain

$$\frac{1}{3g} (W_o + qx + q\Sigma\ell_i + gm_{red})^3 - \frac{1}{2g} [F + \frac{M_F}{r} + q\Sigma\ell_i (1 - \sin \alpha_i)$$

$$+ gm_{red}] (W_o + qx + q\Sigma\ell_i + gm_{red})^2 + \frac{C_o}{2g}$$

$$= \frac{v^2}{2g} (W_o + qx + q\Sigma\ell_i + gm_{red})^2, \tag{1b}$$

where the integration constant may be determined from the condition: $v = 0$ when $x = 0$, i.e.

$$C_o = (W_o + q\Sigma\ell_i + gm_{red})^3 + [F + \frac{M_F}{r} + q\Sigma\ell_i (1 - \sin \alpha_i)$$

$$+ gm_{red}] (W_o + q\Sigma\ell_i + gm_{red})^2.$$

It is convenient to introduce the following abbreviations

$$(W_O + qx + q\Sigma\ell_1 + gm_{red})/W_O = \zeta, \qquad (4)$$

$$(W_O + q\Sigma\ell_1 + gm_{red})/W_O = \zeta_O, \qquad (4a)$$

$$[F + \frac{M_F}{r} + q\Sigma\ell_1(1 - \sin\alpha_1) + gm_{red}]/W_O = \gamma. \qquad (4b)$$

The velocity may now be written as

$$v^2 = \frac{W_O}{q} g \left\{ \frac{2}{3}(\zeta^3 - \zeta_O^3) + \gamma(\zeta_O^2 - \zeta^2) \right\}/\zeta^2. \qquad (1c)$$

With the notation

$$W_O/g = x_O, \text{ i.e. } \frac{dx}{x_O} = d\zeta,$$

we obtain after some simple calculations the time

$$t = \sqrt{6\frac{x_O}{g}} \int_{\zeta_O}^{\zeta} \frac{\zeta d\zeta}{\{4(\zeta^3 - \zeta_O^3) + 6\gamma(\zeta_O^2 - \zeta^2)\}^{1/2}}, \qquad (5)$$

when it is assumed that $\zeta = \zeta_O(x = 0)$ for $t = 0$.

Equation (1) is now integrated, but it is more con-
venient to introduce a new independent variable, let us say u,
and present the kinematic quantities (displacement, velocity,
acceleration) and also the time as dependent on this new variable
u.

4. Introduction of a new independent variable. From Eq.(1b) the
time t may always be expressed by the integral

$$t = \int_{x_O}^{x} \frac{dx}{x} = \int_{\zeta_O}^{\zeta} \frac{d\zeta}{\zeta}, \qquad (6)$$

where $x = x_0$ correspond to $t = 0$. In the second integral we have introduced ζ instead of x.

Instead of expressing the time as a function of the displacement let us introduce a new independent variable

$$u = \int \frac{d\zeta}{\zeta} = \int \frac{d(\ln\zeta)}{\dot{\zeta}} , \qquad (7)$$

the inverse function being

$$\zeta = p(u). \qquad (8)$$

From Eq.(6) we have

$$dt = \frac{d\zeta}{\dot{\zeta}} , \qquad (6a)$$

and from Eq.(7)

$$du = \frac{d\zeta}{\zeta\dot{\zeta}} = \frac{dt}{\zeta} = \frac{dt}{p(u)} . \qquad (7a)$$

Consequently we find, by integrating,

$$t = \int_{u_0}^{u} p(u)du. \qquad (9)$$

The velocity is according to Eqs.(7a) and (8)

$$\dot{\zeta} = \zeta^{-1} \frac{d\zeta}{du} = \frac{p'(u)}{p(u)}. \qquad (10)$$

The acceleration expressed by $p(u)$ is

$$\ddot{\zeta} = \frac{d\dot{\zeta}}{dt} = \frac{d}{dt} \left(\frac{p'(u)}{p(u)}\right) = \frac{d}{du} \frac{p'(u)}{p(u)} \frac{du}{dt} = \frac{p(u)p''(u) - p'(u)^2}{p(u)^3} .$$

As $\ddot{\zeta}$ is given as a function $F(\zeta) = F\left[p(u)\right]$ we get the following differential equation for $p(u)$:

$$\frac{p'(u)}{p(u)} \left[\frac{p''(u)}{p'(u)} - \frac{p'(u)}{p(u)} \right] = F[p(u)] p(u). \tag{11}$$

This equation may serve as a check at the end of the calculation.

In our case we have to substitute

$$u = \int_{\xi}^{\infty} \frac{d\xi}{\left\{ 4(\xi^3 - \xi_0{}^3) + 6\gamma(\xi^2 - \xi_0{}^2) \right\}^{1/2}} \tag{7b}$$

which is an elliptic integral, and the inverse function

$$\xi = p(u) \tag{8a}$$

is an elliptic function of u.

By means of the substitution

$$\xi = (s + \frac{\gamma}{2}), \quad d\xi = ds, \tag{12}$$

we transform the integral (7b) into

$$u = \int_{s}^{\infty} \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}}, \tag{7c}$$

where the invariants of the elliptic integral

$$g_2 = 3\gamma^2, \quad g_3 = 4\xi_0{}^2(\xi_0 - \frac{3}{2}\gamma) + \gamma^3 \tag{13}$$

and the inverse function

$$s = p(u) \tag{8b}$$

is the Weierstrass p-function if we choose the indicated limits of integration in (7b) and (7c).

From Eq.(5) we obtain the time

$$t = (\frac{6x_0}{3})^{1/2} \int_{s_0}^{s} \frac{(s + \frac{\gamma}{2})ds}{\sqrt{4s^3 - g_2 s - g_3}} \qquad (5a)$$

the integral expressing the time

$$\int_{s_0}^{s} \frac{sds}{\sqrt{4s^3 - g_2 s - g_3}} = \int_{u_c}^{u} p(u)du = \zeta(u_0) - \zeta(u) \qquad (5b)$$

where $\zeta(u)$ is the Weierstrass ζ-function.

For the time we finally get

$$t = (\frac{6x_0}{3})^{1/2} [\zeta(u_0) - \zeta(u) + \frac{\gamma}{2} (u - u_0)]. \qquad (5c)$$

From Eqs.(4), (12) and (8b) we have

$$x = x_0 [(p(u) + \frac{\gamma}{2}) - 1] - (\sum_1 \ell_1 + \frac{g}{q} m_{red}) \qquad (8c)$$

and the velocity from Eqs. (1c), (8b) and (12)

$$\dot{x} = \sqrt{\frac{x_0 g}{6}} \frac{p'(u)}{p(u) + \frac{\gamma}{2}} . \qquad (10a)$$

The acceleration can be calculated from Eq.(1a), which may be written

$$W_0 (\zeta - \gamma) = \frac{1}{g} W_0 \frac{d}{dt} (\xi v). \qquad (11a)$$

Recognizing that

$$\frac{d}{dt} (W_0 \xi) v = q v^2,$$

we obtain from Eq. (11a)

$$W_0(\xi - \Upsilon) = \frac{W_0}{g} \xi \frac{dv}{dt} + \frac{a}{g} v^2. \tag{11b}$$

By means of Eq.(8b) and (12) we get

$$a = \frac{dv}{dt} = \frac{g}{p(u) + \frac{\Upsilon}{2}} \left[p(u) - \frac{1}{2} \Upsilon - \frac{1}{6} \frac{p'(u)^2}{(p(u) + \frac{\Upsilon}{2})^2} \right]. \tag{11c}$$

There is an asymptotic value of acceleration equal to $g/3$. This constant value of acceleration corresponds to a straight line for velocity and a parabolic curve for displacement.

By specializing the results obtained above to the case $\Upsilon = 0$ we get the same expressions as in the earlier paper. Therefore we don't need consider this special case now.

5. Investigation of the discriminant of the Weierstrass p-function.

From a numerical calculating point of view it is important to consider the discrimant of the Weierstrass p-function, i.e. [6]

$$\Delta = g_2^3 - 27g_3^2, \tag{14}$$

where g_2, g_3 are the invariants, introduced in Eq.(7c). Inserting g_2 and g_3 from Eq.(13) in (14) we obtain after some elementary calculations

$$\Delta = - 216 \, \xi_0^2 (\xi_0 - \frac{3}{2} \Upsilon)(\xi^3 + 2 \, \xi_0^3 - 3\Upsilon \, \xi_0^2), \tag{14a}$$

where ξ_0 and Υ are given by (4a) and (4b). From Eq.(1a), however, we see that motion is possible only if $\xi_0 > \Upsilon$ and from (4a) and (4b) that $\xi_0 > 0$, $\Upsilon \geq 0$, where $\Upsilon = 0$ corresponds to the case investigated earlier. Writing the discriminant

$$\Delta = 432 \, \zeta_0^6 (\tfrac{3}{2} \tfrac{\Upsilon}{\zeta_0} - 1)(\tfrac{\Upsilon}{\zeta_0} - 1)^2 (\tfrac{\Upsilon}{2\zeta_0} + 1). \tag{14b}$$

The value of the discriminant for $\Upsilon = 0$ is

$$\Delta_0 = - 432 \, \zeta_0^6. \tag{14c}$$

In order to obtain a convenient representation of the discriminant Δ we write

$$\Delta / \Delta_0 = - (\tfrac{3}{2} \tfrac{\Upsilon}{\zeta_0} - 1)(\tfrac{\Upsilon}{\zeta_0} - 1)^2 (\tfrac{\Upsilon}{2\zeta_0} + 1). \tag{14d}$$

This expression considered as a function of Υ / ζ_0 has a maximum for $\Upsilon / \zeta_0 = (\sqrt{17} - 1)/4 = 0.781$ and is equal to zero for $\Upsilon / \zeta_0 = 2/3$ and $\Upsilon / \zeta_0 = 1$. The curve Δ / Δ_0 is shown in **Fig. 5**.

If we write the polynomial under the square root of Eq.(7c)

$$4s^3 - g_2 s - g_3 = 4(s - e_1)(s - e_2)(s - e_3), \tag{15}$$

e_1, e_2, e_3 being the roots of the equation

$$4s^3 - g_2 s - g_3 = 0 \tag{15a}$$

the negative value of the discriminant Δ means that two of the roots, let us say, e_1 and e_3 are complex, the third root e_2 being real.

Introducing the quantities H and k^2 defined by

$$H^2 = (e_2 - e_1)(e_2 - e_3) = 2e_2^2 + \frac{g_3}{4e_2}$$

$$k^2 = \tfrac{1}{2} - \frac{3e_2}{4H}, \tag{16}$$

where k is the modulus of the Jacobian elliptic functions, Eqs. (16) establish the connections between the quantities g_3, e_2, H and k^2, where $g_3 = 3\gamma^2$ and e_2 may be found as the real root of Eq.(15a). To get a survey over the numerical calculations it is, however, more convenient to consider the values H and k^2 as given and from these calculate g_3 and e_2, getting

$$e_2 = \frac{2}{3} H(1 - 2k^2),$$

$$g_3 = (\frac{2H}{3})^3(1 - 2k^2)\,[9 - 8(1 - 2k^2)^2]\,. \qquad \left.\right\} \quad (16a)$$

The connection between the invariant g_2 and g_3, on the other hand, are given by

$$g_2 = 4e_2^2 - \frac{g_3}{e_2} = 3\gamma^2\,. \qquad (17)$$

Introducing e_2 and g_3 from Eqs.(16a), we obtain

$$g_2 = 4H^2[\,\frac{4}{3}(1 - 2k^2) - 1]\,. \qquad (17a)$$

Because of $g_2 = 3\gamma^2$, we must have $g_2 \geq 0$ or

$$\frac{4}{3}(1 - 2k^2)^2 - 1 > 0,$$

which means $k^2 \leq 0.06699$, $k \leq 0.259$, the sign equal corresponding to the equianharmonic case ($g_2 = 0$) considered earlier. Because of $g_2 > 0$ the value of k^2 is small and therefore as a good approximation we may assume $k^2 = 0$, giving

$$e_2 = \frac{2}{3} H = \frac{1}{3} g_2 = \Upsilon,$$

$$\left.\begin{array}{c} \\ \\ \end{array}\right\} \qquad (16b)$$

$$g_3 = (\frac{2}{3} H)^3 = \Upsilon^3.$$

On the other hand we have from Eq.(13)

$$g_3 = 4\xi_0^2(\xi_0 - \frac{3}{2} \Upsilon) + \Upsilon^3 \qquad (13a)$$

and therefore the following condition between ξ_0 and Υ holds for the case $k^2 = 0$:

$$\Upsilon/\xi_0 = 2/3,$$

all quantities being expressed by Υ.

As $\Upsilon/\xi_0 = 2/3$ corresponds to the special case $\Delta = 0$, where

$$4s^3 - g_2 s - g_3 = 4s^3 - 3\Upsilon^2 s - \Upsilon^3 = 0$$

with all roots real $e_1 = \Upsilon$, $e_2 = e_3 = -\Upsilon/2$, we have the following equation for the Weierstrass p-function

$$p(u) = e_3 + \frac{e_1 - e_3}{sn^2(u \sqrt{e_1 - e_3})}, \qquad (18)$$

the real half period being

$$\omega_1 = \frac{k}{\sqrt{e_1 - e_3}} = \frac{\pi}{\sqrt{6\Upsilon}} = \frac{\pi}{\sqrt{6e_1}}. \qquad (19)$$

6. <u>Representation of the kinematic quantities in the case</u> <u>$\Upsilon/\xi_0 = 2/3$.</u>

From Eq.(18) we get, inserting for e_1 and e_3

$$p(u) + \frac{\gamma}{2} = \frac{\frac{3}{2}\gamma}{\sin^2(u\sqrt{\frac{3}{2}\gamma})} \ .$$

The value of u corresponding to the argument $\pi/2$ is $u_0 = \frac{\frac{\pi}{2}}{\sqrt{\frac{3}{2}\gamma}}$

and therefore we may write

$$p(u) + \frac{\gamma}{2} = \frac{\frac{3}{2}\gamma}{\sin^2(\frac{\pi}{2}\frac{u}{u_0})} \ .$$

Introducing this in Eq.(8c) we get, remembering from Eq.(4a) that

$$\Sigma \ell_1 + \frac{g}{q}m_{red} = \frac{W_0}{q}(\zeta_0 - 1) = x_0(\frac{3}{2}\gamma - 1),$$

(8d)

$$x = x_0 \frac{3}{2}\gamma \left[\frac{1}{\sin^2(u\sqrt{\frac{3}{2}\gamma})} - 1\right].$$

From this we see that the boundary condition x = 0 for t = 0 ($u = u_0$) is satisfied. Equation (8d) may be written

$$x = \frac{3}{2}\gamma\, x_0 \cot^2(u\sqrt{\frac{3}{2}\gamma}).$$

(8e)

The time may be written as

$$t = (\frac{6x_0}{g})^{1/2}\int [p(u) + \frac{\gamma}{2}]\,du = (\frac{6x_0}{g})^{1/2}\frac{3}{2}\gamma\int\frac{du}{\sin^2(\sqrt{\frac{3}{2}\gamma}u)}$$

(5d)

which, after integration, yields

$$t = - (\frac{6x_0}{g})^{1/2} (\frac{3}{2}\gamma)^{1/2} \cot (\sqrt{\frac{3}{2}\gamma} u). \tag{5e}$$

From this we obtain

$$\frac{3}{2}\gamma \cot^2(\sqrt{\frac{3}{2}\gamma} u) = \frac{gt^2}{6x_0} \tag{5f}$$

which inserted in Eq. (8e) yields the simple result

$$x = \frac{g}{6} t^2. \tag{8f}$$

From Eq.(10a) for the velocity we get

$$\dot{x} = - 2\sqrt{\frac{x_0 g}{6}} (\frac{3}{2}\gamma)^{1/2} \cot \sqrt{\frac{3}{2}\gamma} u =$$

$$2(\frac{6x_0}{g})^{1/2} (\frac{6x_0}{g})^{-1/2} t \tag{10b}$$

or, in agreement with (8f),

$$\dot{x} = \frac{g}{3} t. \tag{10c}$$

In the same manner we finally obtain for the acceleration

$$a = \frac{2}{3}\frac{g}{\gamma} \sin^2(\sqrt{\frac{3}{2}\gamma} u) \left[\frac{\frac{3}{2}\gamma}{\sin^2(\sqrt{\frac{3}{2}\gamma} u} - \gamma - \gamma \frac{\cos^2(\sqrt{\frac{3}{2}\gamma} u)}{\sin^2(\sqrt{\frac{3}{2}\gamma} u)} \right]$$

$$a = \frac{2}{3}\frac{g}{\gamma} \sin^2(\sqrt{\frac{3}{2}\gamma} u) [\frac{3}{2}\gamma - \gamma]\frac{1}{\sin^2(\sqrt{\frac{3}{2}\gamma} u)} = \frac{g}{3}.$$

We therefore have for $\xi_0 = \frac{2}{3}\gamma$ the very simple case of constant acceleration, velocity linearly dependent on time and displacement depending on the square of time. If we are starting with an acceleration equal to $g/3$, the motion will continue with this constant value of acceleration. In Fig. 6 is shown the displacement

as a function of time for the cases $\Upsilon/\zeta_o = 0$, $\Upsilon/\zeta_o = 1/3$ and $\Upsilon/\zeta_o = 2/3$ (constant acceleration); Fig. 7 and 8 present velocity resp. acceleration for the same values of Υ/ζ_o.

7. <u>Simple proof that constant acceleration is equal to $g/3$</u>. The equation of motion may be written

$$\frac{dm}{dt} v + m \frac{dv}{dt} = P, \tag{18}$$

where $dm = qdx/g$, $dm/dt = qv/g$,

$$\frac{dm}{dt} v = \frac{q}{g} v^2. \tag{19}$$

At the beginning of motion we have $v = 0$ and

$$\frac{dv}{dt} = \frac{dv}{dt}_{(t=0)} = a, \quad P_{t=0} = m_o \frac{dv}{dt}_{(t=0)} = m_o a. \tag{18a}$$

Substracting this last equation from Eq.(18) we get

$$\frac{q}{g} v^2 + (m - m_o) a = P - P_{(t=0)} = qx \tag{18b}$$

because the difference of forces at an arbitrary time t and $t = 0$ is qx. Furthermore is $m - m_o = \frac{q}{g} x$. By constant acceleration a the square of velocity is $v^2 = 2ax$, which value inserted in Eq. (18b) yields

$$\frac{q}{g} 2ax + \frac{q}{g} ax = qx \text{ or } a = \frac{g}{3} = \text{const.}$$

This is the only value of constant acceleration which is valid for all values of x, that means during the whole motion from beginning to stop. From Eq.(18b) we see that a constant acceleration is possible by motion with variable mass if the mass in

motion and acting forces are increasing proportional to the displacement. If the motion takes place under the influence of gravity the constant value of acceleration is just equal to $g/3$.

8. <u>Summary</u>. The motion of an anchor chain under the influence of gravity and friction forces is considered. The equation of motion involves also the influence of the moment of inertia of the windlass and the frictional moment acting on the axis of the windlass. The solution of the equation can be expressed by the elliptic functions of Weierstrass, the time by the ζ- function of a parameter υ, the kinematic quantities (displacement, velocity, acceleration) by the p-function and its derivative of the same parameter. In a special case these functions can be expressed by trigonometric functions, corresponding to a constant acceleration $a = g/3$, the velocity $v = gt/3$ and the displacement $s = gt^2/6$. The absence of frictional forces corresponds to the equianharmonic case of the Weierstrass function, tabulated by A. G. Greenhill [7]. All real cases of motion of the anchor chain are included between this case and the case of constant acceleration $a = g/3$ and can be found by linear interpolation with respect to the parameter Υ, expressing the ratio of the friction forces and the accelerating forces.

9. <u>Appendix</u>. As mentioned in section 3 it is possible to give a more complete expression of the frictional force. This will be done in the following. Let us consider the forces acting on an inclined plane $(n - k)$ with the angle of the slope equal to $\alpha_{(n-k)}$, the length $\ell_{(n-k)}$ and the friction coefficient μ, which

may be assumed equal for all inclined planes (Fig.9). For the
motion of this part of the chain we have the following equation

$$S_{n-k} - S'_{n-k} + g\ell_{n-k} (\sin \alpha_{h-k} - \mu \cos \alpha_{n-k}) = \frac{q}{g} \ell_{n-k} \frac{dv}{dt}. \quad (20)$$

At the corner of two adjacent slopes there exists the following
relation between the forces in the chain on both sides of the
corner

$$S'_{n-k+1} = S_{n-k} \exp [\mu (\alpha_{n-k+1} - \alpha_{n-k})]. \quad (21)$$

The prime index indicates the force in the chain just
below the corner, the same notation without the prime index in-
dicates the force on the same slope just above the corner. If
the first inclined plane where the chain is moving after leaving
the windlass has the number (n - m), the force in the chain is

$$\left. \begin{aligned} S'_{n-m} &= \frac{d}{dt}(m_{red} \, v) + \frac{M_F}{r} + \frac{q}{g}(\ell_0 + \ell_1 + \ell_2)\frac{dv}{dt} \\ &\quad - q(\ell_0 \sin \alpha_0 + \ell_1 \sin \alpha_1 + \ell_2 \sin \alpha_2), \end{aligned} \right\} \quad (22)$$

ℓ_0, ℓ_1, ℓ_2 are the lengths of the chain of the three first sec-
tions, according to Fig. 4. On the other hand the force in the
chain just below the last corner corresponding to the vertical
plane (n + 1), is given by

$$S'_{n+1} = (W_0 + qx) - \frac{1}{g} \frac{d}{dt} [(W_0 + qx)v]. \quad (23)$$

From the last four equations we can establish an equation of

motion similar to Eq.(1a), section 3. It is convenient to introduce

$$L = (\ell_0 + \ell_1 + \ell_2) \exp \left[(\frac{\pi}{2} = \alpha_{n-m}) \right]$$

$$+ \sum_{k=0}^{k=m} \ell_{n-k} \exp \left[\mu(\frac{\pi}{2} - \alpha_{(n-k)}) \right],$$

$$L' = \sum_{k=0}^{k=m} \ell_{(n-k)}(\sin \alpha_{n-k} - \mu \cos \alpha_{n-k}) \exp \left[\mu(\frac{\pi}{2} - \alpha_{n-k}) \right].$$

(24)

If we expand the equation (23) on both sides by $(gm_{red} + qL)$, in order to have a total differential we obtain the equation of motion

$$(W_0 + qx + gm_{red} + qL) - \frac{1}{3} \frac{d}{dt} \left[(W_0 + qx + gm_{red} + qL) \, v \right]$$

$$= gm_{red} + gL - \left\{ \frac{M_T}{r} \exp \left[\mu(\frac{\pi}{2} - \alpha_{n-m}) \right] + qL' \right\}.$$

(25)

The **two** last terms contain the influence of the friction forces on the system. If we now multiply the equation by

$$(W_0 + qx + gm_{red} + qL) \, vdt = (W_0 + qx + gm_{red} + qL) \, dx$$

and integrate, we get the same equation as Eq.(1b), section 3, apart a slight change of some of the constants. We thus have given a proof that it is allowed to introduce the friction as a constant, if we neglect the influence of centrifugal forces at the corners during motion.

Bibliography

1. R. Gran Olsson, Quarterly of Applied Mathematics, vol. VII
                (1950), p.p. 463-470.

2. T.S. Brown, Institute of Marine Engineers Transaction, vol.
                58(1946), pp. 1-16 or Bulletin Technique du
                "Veritas", vol. 30(1948), pp. 53-62.

3. S. Timoshenko and D.H. Young, Advanced Dyamics, McGraw-Hill
                Book Co., Inc., New York 1948, pp. 114-116.

4. E.H. Neville, Jacobian Elliptic Functions, 2nd ed., Claredon
                Press, Oxford 1951, pp. 26-49.

5. A. Sommerfeld, Lectures on Theoretical Physics, vol. I,
                Mechanics, Academic Press Inc., New York
                1952, pp. 62-66.

6. L.M. Milne-Thomson, Jacobian Elliptic Function Tables,
                Dover Publications, Inc., New York 1950,
                pp. 23-24.

7. A.G. Greenhill, Proc. Roy. Artillery Inst. 17, pp. 181-226
                (1889); or E. Jahnke and F. Emde, Tables of
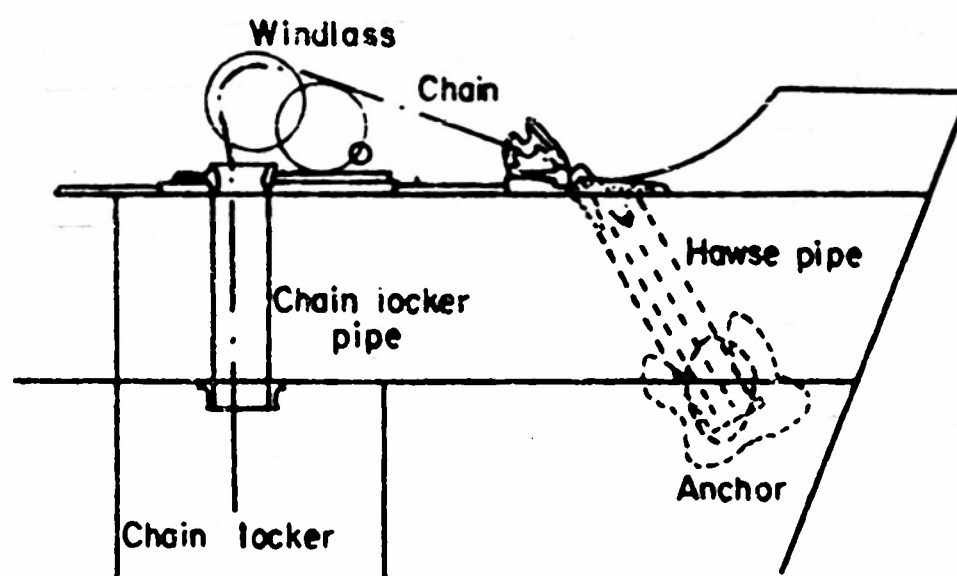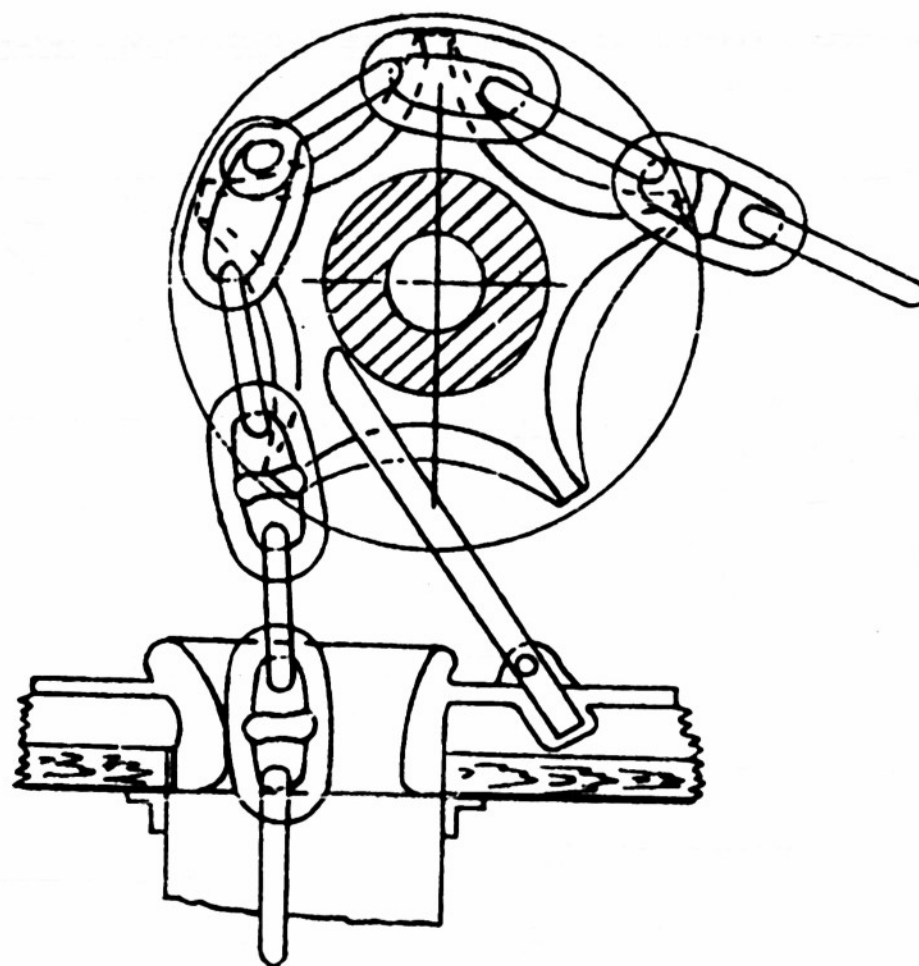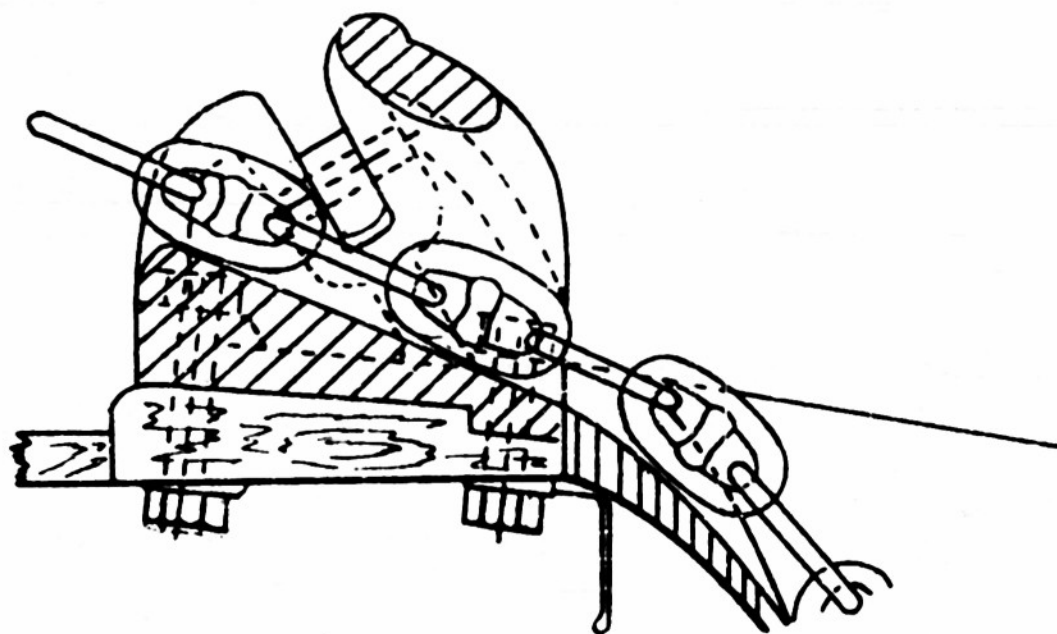   Functions, Dover Publication, New York 1945,
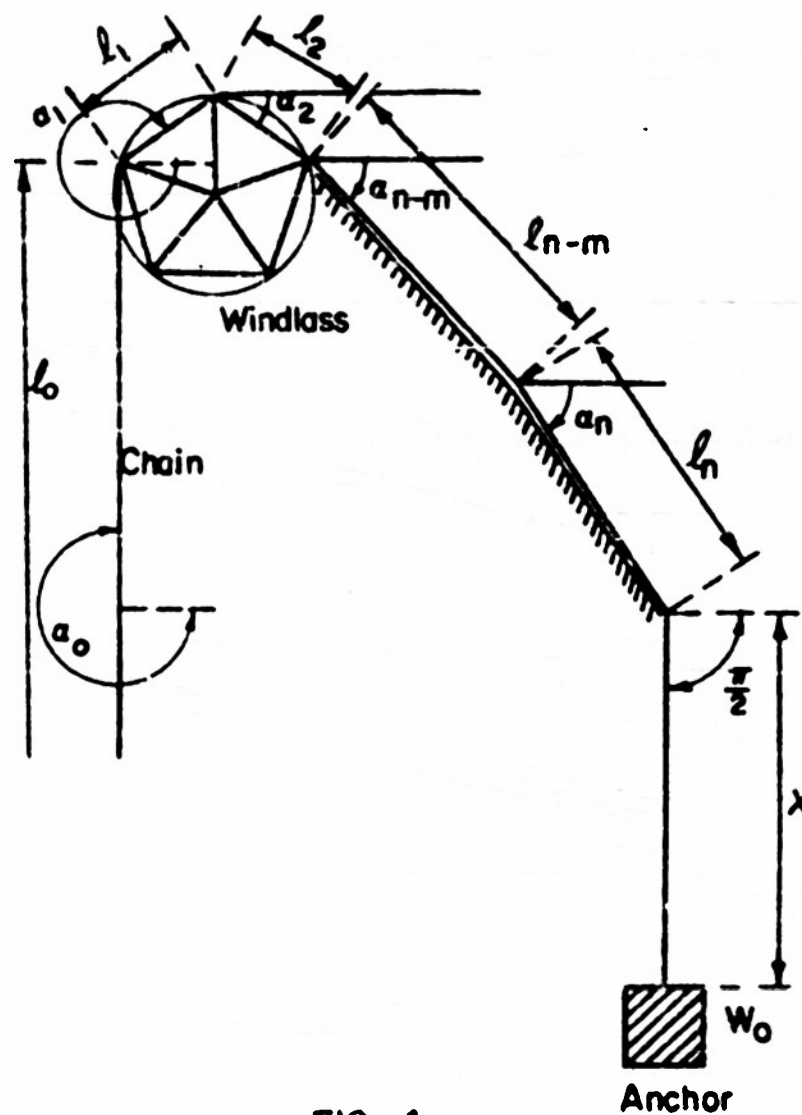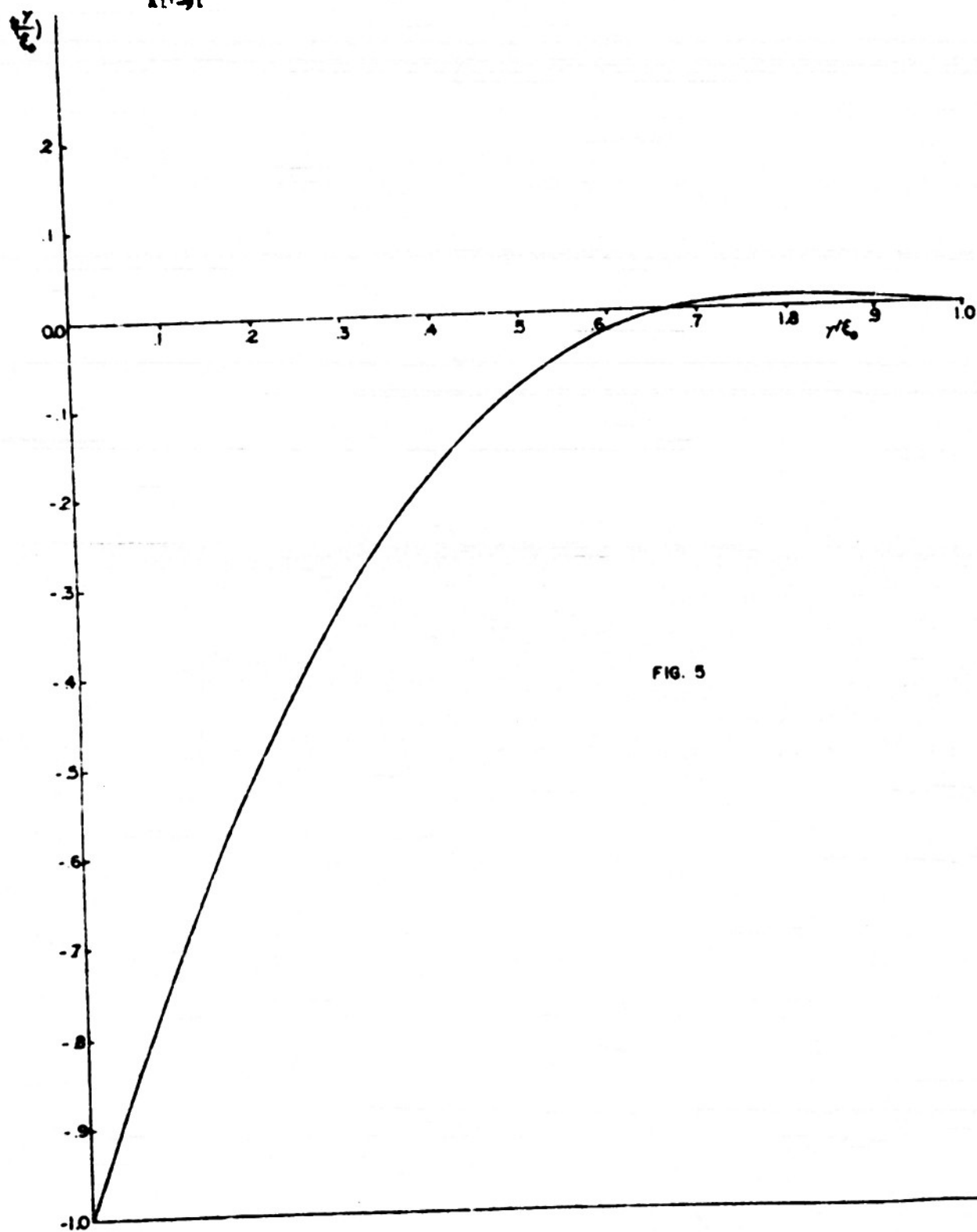   pp. 102-104.

FIG. 1

FIG. 2

FIG. 3

FIG. 4

FIG. 5
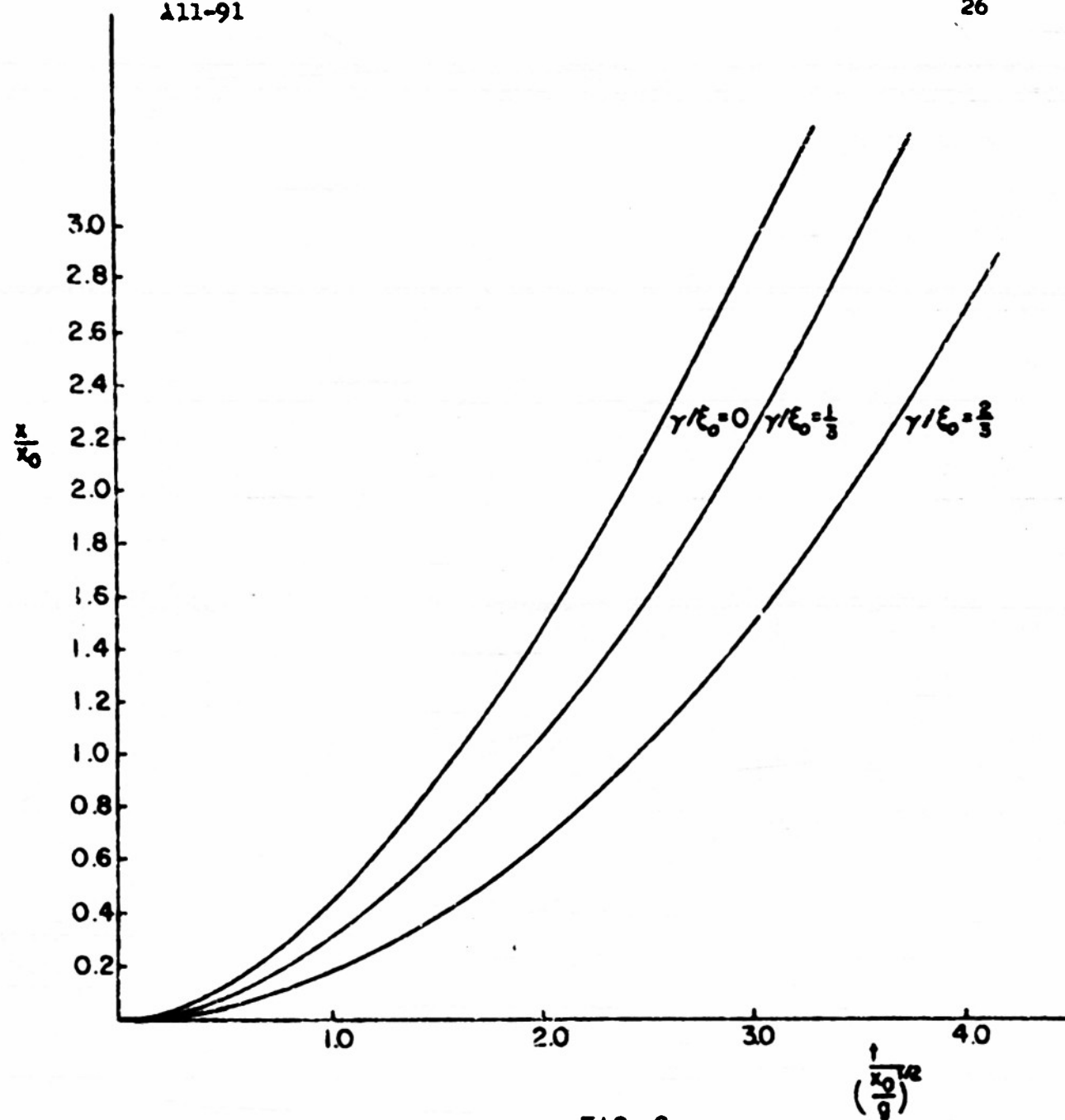
FIG. 6
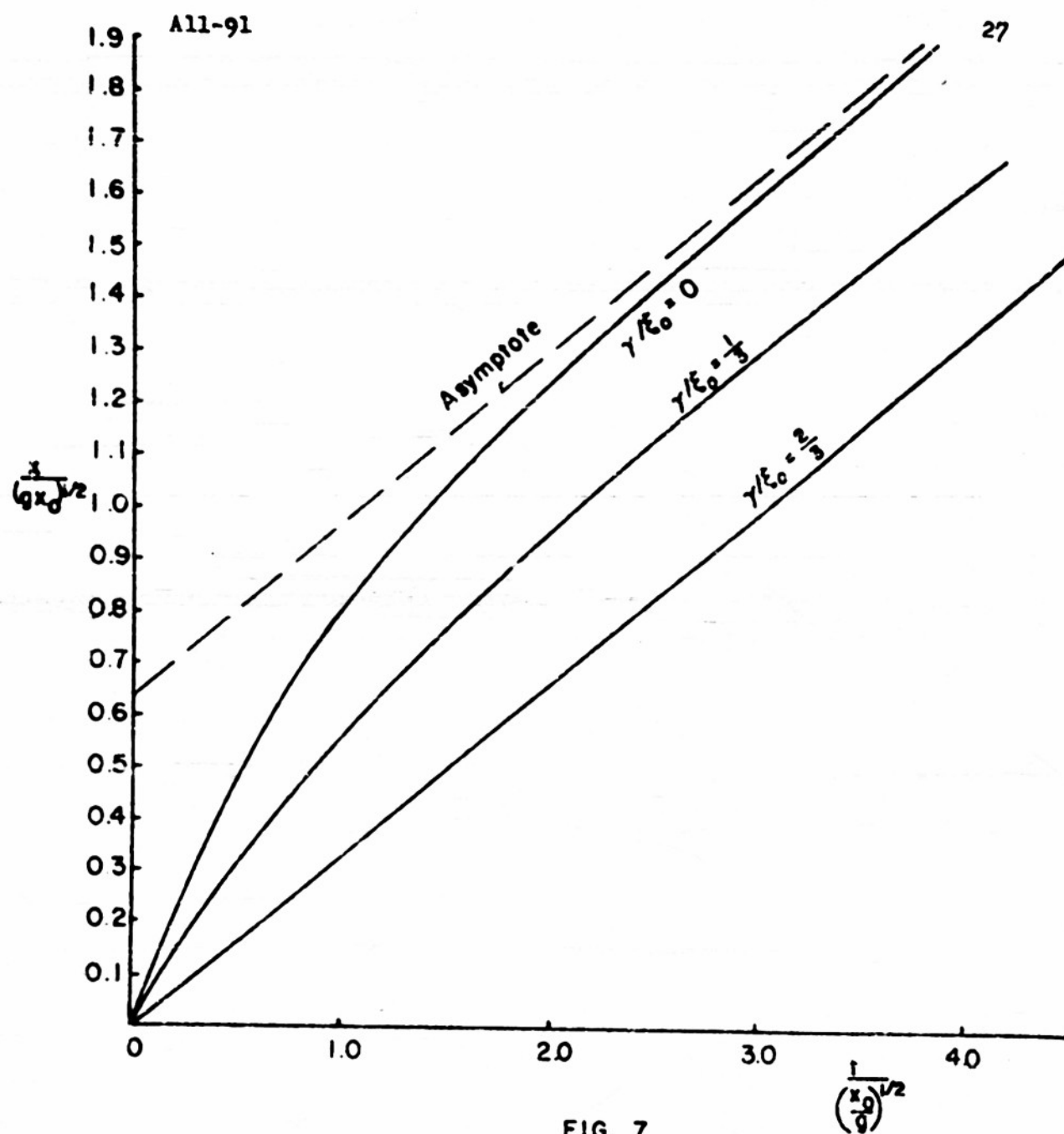
FIG. 7
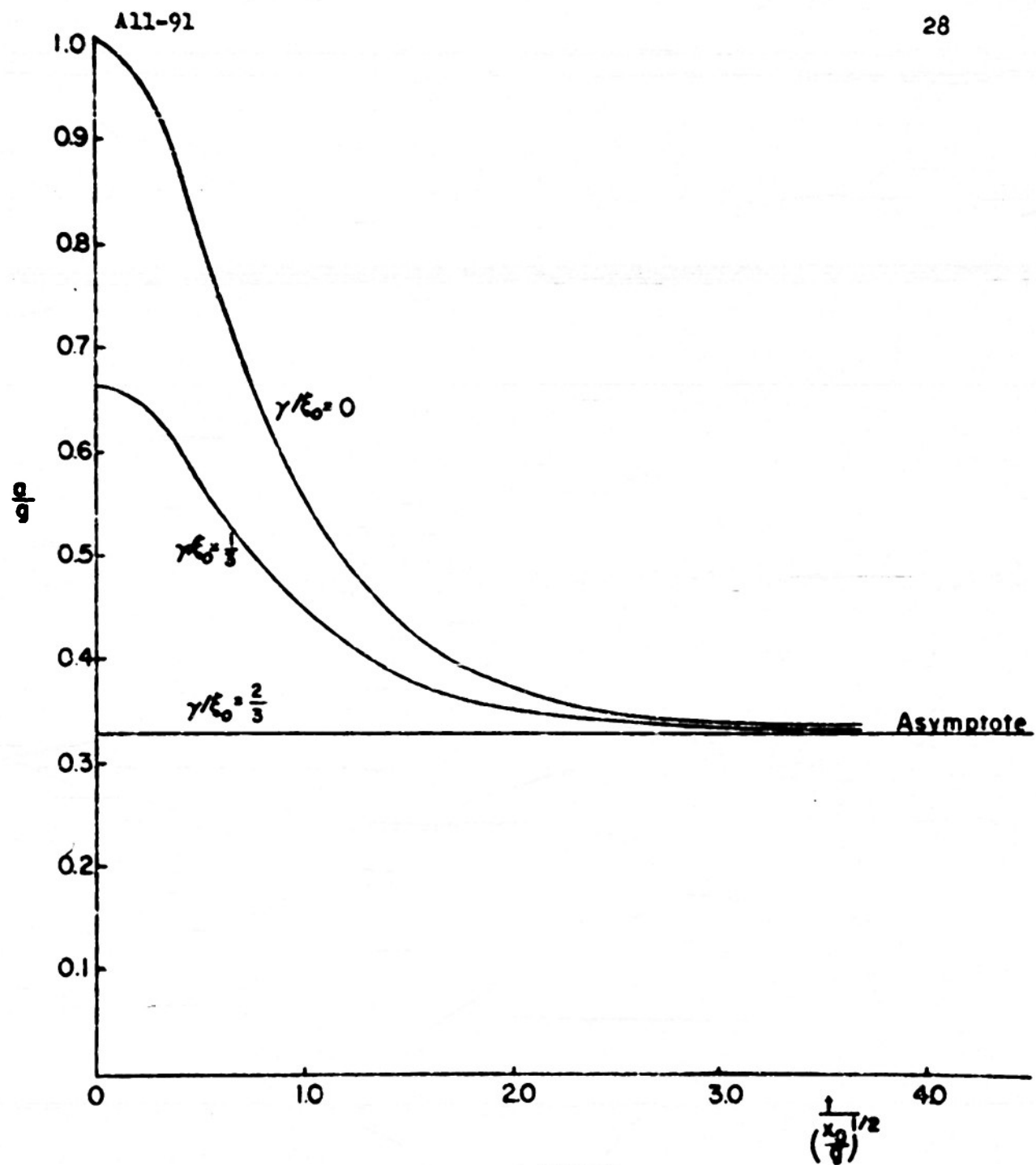
FIG. 8
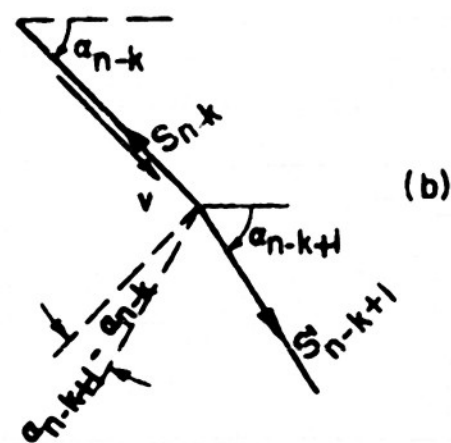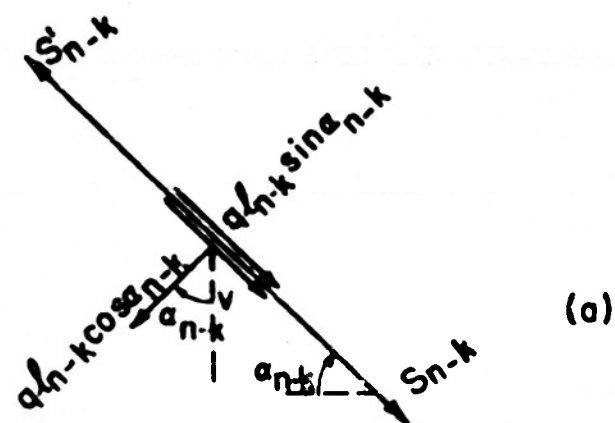
FIG. 9

Issued by

BROWN UNIVERSITY
Contract N7onr-358, T. O. 1
NR 041 032

Office of Naval Research
Washington 25, D. C.
M-2   Attn: Mathematics Branch (Code 432)
M-1   Mechanics Branch (Code 438)
M-1   PhysicalBranch   (Code 421)
M-1   Metallurgy Branch (Code 423)

M-2   Commanding Officer
Office of Naval Research Branch Office
150 Causeway Street
Boston 14, Massachusetts

M-1   Commanding Officer
Office of Naval Research Branch Office
346 Broadway
New York, New York

M-1   Commanding Officer
Office of Naval Research Branch Office
844 North Rush Street
Chicago 11, Illinois

M-1   Commanding Officer
Office of Naval Research Branch Office
1000 Geary Street
San Francisco 9, California

M-1   Commanding Officer
Office of Naval Research Branch Office
1030 East Green Street
Pasadena 1, California

M-17   Officer-in-Charge
Office of Naval Research
Navy #100
Fleet Post Office
New York, New York

M-9   Director
Naval Research Laboratory
Washington 20, D. C.
    Attn: Scientific Information Division
M-2   Library (Code 2021)
M-1   Applied Mathematics Branch (Code 3830)
M-1   Shock and Vibrations Section (Code 3850)
M-1   Structures Branch (Code 3860)

Bureau of Ships
Department of the Navy
Washington 25, D. C.
M-2   Attn: Code 364 (Technical Library)
R-1   Code 423 (Underwater Explosion Research)
M-1   Code 442 (Scientific Section, Design)

David Taylor Model Basin
Carderock, Maryland
M-2   Attn: Library
M-1   Structural Mechanics Division

Naval Ordnance Laboratory
White Oak, Silver Spring 19, Maryland
M-2   Attn: Library

Bureau of Aeronautics
Department of the Navy
Washington 25, D. C.
M-1   Attn: AER-TD-414
R-1   Materials Branch
R-1   Design Elements Division

Bureau of Yards and Docks
Department of the Navy
Washington 25, D. C.
R-2   Attn: Director, Research Division

Commander
Norfolk Naval Shipyard
Norfolk, Virginia
M-1   Attn: Technical Library (Code 243A)
M-1   UERD   (Code 290)

Superintendent
Aeronautical Structures Laboratory
Building 600, Naval Air Experimental Station
Philadelphia 12, Pennsylvania
R-1   Attn: Experimental Structures Section

Office, Assistant Chief of Staff, G4
The Pentagon
Washington, D. C.
M-1   Attn: Research and Development Division

M-1   The Chief, Armed Forces Special Weapons Project
Department of Defense
P. O. Box 2610
Washington, D. C.

U. S. Army Arsenal
Watertown 72, Massachusetts
M-1   Attn: Dr. R. Beeuwkes

Frankford Arsenal
Pitman-Dunn Laboratory
Philadelphia 37,
M-1   Attn: Dr. Herbert I. Fusfold

Picatinny Arsenal
Dover, New Jersey
M-1   Attn: Mr. L. Gilman

Commanding General
Air Materiel Command
Wright-Patterson Air Force Base
Dayton, Ohio
M-1   Attn: Chief, Materials Division (DCRTS)
R-1   Head, Structures Lab (MCREX-B)

Department of Commerce
Office of Technical Service
Washington 25, D. C.
M-1   Attn: Library Section

National Advisory Committee for Aeronautics
1724 F. Street NW
Washington 25, D. C.
M-1   Attn: Chief of Aeronautical Intelligence

National Advisory Committee for Aeronautics
Langley Aeronautical Laboratory
Langley Field, Virginia
M-1   Attn: Library

National Advisory Committee for Aeronautics
Lewis Flight Propulsion Laboratory
Cleveland Airport
Cleveland 11, Ohio
M-1   Attn: Library

National Bureau of Standards
Washington, D. C.
M-1   Attn: Dr. W. H. Ramberg

Director of Research
Sandia Corporation
Albuquerque, New Mexico
M-1   Attn: Dr. R. P. Peterson

Brooklyn Polytechnic Institute
85 Livingston Street
Brooklyn, New York
R-1   Attn: Dr. N. J. Hoff
R-1   Dr. H. Reissner
M-1   Dr. F. S. Shaw (Dept. Aero. Engrg. & Appl. Mech.)

Brown University
Providence 12, Rhode Island
M-1   Attn: Chairman, Graduate Division of Applied Mathematics

California Institute of Technology
Pasadena, California
R-1   Attn: Dr. J. G. Kirkwood
R-1   Dr. Pol Duwez

University of California
Berkeley, California
M-1   Attn: Dr. J. E. Dorn
R-1   Dr. H. Hultgren
R-1   Dr. G. C. Evans
M-1   Dr. C. F. Garland

University of California
Los Angeles, California
R-1   Attn: Dr. I. S. Sokolnikoff
R-1   Dr. L. Rosenthal

Carnegie Institute of Technology
Pittsburgh, Pennsylvania
R-1   Attn: Dr. J. S. Koehler
R-1   Dr. G. H. Handelman
M-1   Dr. E. Saibel
R-1   Dr. H. J. Greenberg
R-1   Dr. E. D'Appolonia

Case Institute of Technology
Cleveland, Ohio
M-1   Attn: Dr. W. M. Baldwin, Jr., Metals Research Laboratory
R-1   Dr. O. Hoffman

Catholic University of America
Washington, D. C.
M-1   Attn: Dr. F. A. Biberstein
R-1   Dr. K. Hertzfeld

University of Chicago
Chicago, Illinois
R-1   Attn: Dr. T. S. Ke
R-1   Dr. C. S. Barrett

Columbia University
New York, New York
M-1   Attn: Dr. R. D. Mindlin
M-1   Dr. H. Bleich

Cornell University
Ithaca, New York
R-1   Attn: Dr. H. S. Sack
R-1   Dr. A. Kantrowitz

University of Florida
Gainesville, Florida
M-1   Attn: Dr. C. G. Smith

Harvard University
Cambridge 38, Massachusetts
R-1   Attn: Dr. R. von Mises
R-1   Dr. F. Birch, Dunbar Laboratory
R-1   Dr. H. M. Westergaard

Illinois Institute of Technology
Chicago, Illinois
R-1   Attn: Dr. L. H. Donnell
R-1   Dr. L. van Griffis
M-1   Dr. E. Sternberg
R-1   Dr. W. Osgood
M-1   Dr. C. A. Eringen

University of Illinois
Urbana, Illinois
M-1   Attn: Dr. N. M. Newmark
R-1   Engineering
R-1   T. J. Dolan
R-1   Dr. F. Seitz, Department of Physics
M-1   Department of Theoretical and Applied Mathematics

Indiana University
Bloomington, Indiana
M-1   Attn: Dr. T. Y. Thomas

Institute for Advanced Study
Princeton, New Jersey
R-1   Attn: Dr. J. von Neumann

Iowa State College
Ames, Iowa
R-1   Attn: Dr. G. Murphy
R-1   Dr. D. L. Hall

Johns Hopkins University
Baltimore, Maryland
M-1   Attn: Dr. W. H. Hoppman, II

M-1   Director, Applied Physics Laboratory
        Johns Hopkins University
        8621 Georgia Avenue
        Silver Spring, Maryland

Lehigh University
Bethlehem, Pennsylvania
R-1   Attn: Mr. Lynn S. Beedle

Massachusetts Institute of Technology
Cambridge 39, Massachusetts
R-1   Attn: Dr. F. B. Hildebrand
R-1   Dr. C. W. MacGregor
R-1   Dr. J. M. Lessels
R-1   Dr. W. M. Murray
R-1   Dr. E. Reissner
R-1   Dr. H. S. Tsien
R-1   Dr. M. Cohen, Rm. 8-413, Department of Metallurgy
R-1   Dr. B. L. Averbach, Department of Metallurgy
R-1   Dr. J. T. Norton
R-1   Dr. E. Orowan
M-1   Dr. R. Bisplinghoff, Dept. Aero. Engr.

University of Michigan
Ann Arbor, Michigan
M-1   Attn: Dr. Bruce G. Johnston
M-1   Dr. Paul Nagdhi
R-1   Dr. N. Coburn
R-1   Dr. W. Kaplan

New York University
Institute for Mathematics & Mechanics
45 Fourth Avenue
New York 3, New York
R-1   Attn: Professor R. Courant
R-1   Dr. G. Hudson

New York University
New York 53, New York
R-1   Attn: Dr. C. T. Wang, Department of Aeronautics

Northwestern University
Evanston, Illinois
R-1   Attn: Dr. M. M. Hetenyi

University of Notre Dame
Notre Dame, Indiana
R-1   Attn: Dr. P. A. Beck

Ohio State University
Columbus, Ohio
M-1   Attn: Dr. B. A. Boley

Pennsylvania State College
State College, Pennsylvania
R-1   Attn: Dr. M. Gensamer
R-1   Dr. J. A. Sauer
R-1   Dr. Joseph Marin
R-1   Dr. J. W. Fredrickson

Princeton University
Princeton, New Jersey
R-1   Attn: Dr. S. Lefschetz
R-1   Dr. L. Lees
R-1   Dr. J. V. Charyk

Rensselaer Polytechnic Institute
Troy, New York
R-1   Attn: Library
R-1   Dr. Paul Leiber

Santa Clara University
Santa Clara, California
M-1   Attn: Dr. R. M. Hermes

Stanford University
Stanford, California
R-1   Attn: Dr. L. Jacobsen
M-1   Dr. A. Phillips, Department of Mechanical Engineering
R-1   Dr. J. N. Goodier

6

Stevens Institute of Technology
Hoboken, New Jersey
R-1   Attn: Dr. E. G. Schneider

Swarthmore College
Swarthmore, Pennsylvania
M-1   Attn: Capt. W. P. Roop

University of Texas
Austin 12, Texas
R-1   Attn: Dr. A. A. Topractsoglou

University of Utah
Salt Lake City, Utah
M-1   Attn: Dr. H. Eyring

Washington State College
Pullman, Washington
R-1   Attn: Dr. B. Fried

Wheaton College
Norton, Massachusetts
R-1   Attn: Dr. H. Geiringer

Aerojet, Inc.
Azusa, California
R-1   Attn: F. Zwicky

Aluminum Company of America
New Kensington, Pennsylvania
M-1   Attn: R. L. Templin

Armstrong Cork Company
Lancaster, Pennsylvania
R-1   Attn: J. W. Scott

Bell Telephone Laboratories
Murray Hill, New Jersey
R-1   Attn: C. Herring
R-1   J. M. Richardson
R-1   D. P. Ling
R-1   W. P. Mason

Corning Glass Company
Corning, New York
R-1   Attn: J. T. Littleton

E. I. Dupont de Nemours & Co., Inc.
Wilmington 98, Delaware
R-1   Attn: J. H. Faupel, Materials of Construction Section

General Electric Company
Schenectady, New York
R-1   Attn: H. Fehr
R-1   H. Poritsky
R-1   J. H. Hollomon

7

General Motors
Detroit, Michigan
R-1   Attn: J. O. Almen

Lockheed Aircraft Company
Department 72-25, Factory A-1, Building 66
Burbank, California
R-1   Attn: Engineering Library

Midwest Research Institute
Kansas City, Missouri
R-1   Attn: C. O. Dohrenwend
R-1   M. Golan

Pratt & Whitney Aircraft Corporation
East Hartford, Connecticut
R-1   Attn: R. Morrison

U. S. Rubber Company
Passaic, New Jersey
R-1   Attn: H. Smallwood

Welding Research Council
Engineering Foundation
29 West 39th Street
New York 18, New York
M-1   Attn: W. Spraragen, Director

Westinghouse Research Laboratories
East Pittsburgh, Pennsylvania
M-1   Attn: Dr. A. Nadai
R-1   Dr. E. A. Davis

Westinghouse Electric Corporation
Lester Branch P. O.
Philadelphia, Pennsylvania
R-1   Attn: R. P. Kroon, Mgr. of Engineering, AGT Division